Robust Stability Analysis of a Class of Smith Predictor-based Congestion Control Algorithms for Computer Networks

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Outline

1. Introduction
2. SP congestion control
3. Review of the geometric approach
4. SCC of the congestion control model
5. Robust stability
The Internet is a relevant example of time-delay system due to propagation of information over communication links. When packets arrive at a rate above the capacity of the output link, the router queue builds and congestion arises. TCP Congestion control is a building block of the Internet designed to avoid congestion and preserve network stability.
TCP Congestion Control Models

The literature dealing with congestion control algorithms modeling is very broad:

- **Stationary models**: predict the average throughput based on average measures such as round trip time, packet loss ratio
  - SQRT Formula (*Mathis et al., 1997*)
  - PFTK formula (*Padhye et al., 1998*)

- **Fluid Models**: based on differential equations
  - *Hollot et al. (2001)*: non-linear differential equations of congestion window $w(t)$ and queue length $q(t)$
  - *Mascolo (1999)*: linear model comprising two time delays, an integrator and a Smith Predictor.

- **Hybrid Models**: time continuous differential equations coupled with discrete dynamics (packet loss events, etc)
  - *Hespana et al. (2001)*: describes all the phases of TCP congestion control (slow start, cong. avoid., fast retx)
Smith-predictor based Congestion Control
(S. Mascolo, 1999)

\[ w(t) \rightarrow k \rightarrow e^{-sT_1} \rightarrow \frac{1}{s} \rightarrow q(t) \]

\[ e^{-sT_2} \]

\[ w(t) \]: Set-point (congestion window)
\[ q(t) \]: Queue length
\[ T_1, T_2 \]: source-destination and destination-source delays

\[ k \]: Controller gain
\[ 1/s \]: bottleneck link queue model
\[ b(t) \]: bottleneck available bandwidth
Why using a Smith Predictor plus proportional controller?

- If the delay \( T_1 + T_2 \) (RTT, round trip time) is exactly known, the closed loop dynamics is that of a first order system (no overshoots, system is always stable)
- A unique parameter to tune having a direct influence on the step response of the system
- It models the TCP congestion control and its variants by proper input shaping of the congestion window \( w(t) \)
- A rate-based congestion control algorithm has been designed and implemented (Grieco and Mascolo, 2004)

Focus of the paper

Smith-predictor is known to be sensitive to delay model uncertainties. What’s the effect of a measurement error in the delay on the stability of the system?
Stability crossing curves for systems with two delays 
(Gu, Niculescu, Chen, 2005)

Let us consider:
- LTI SISO system \( G(s) = G_0(s)e^{-\tau s} \), \( G_0(s) \) delay free plant
- Smith predictor controller \( C(s) \), nominal delay \( \bar{\tau} = T_1 + T_2 \),
  \( \Delta \) delay uncertainty

Characteristic equation:

\[
1 - h(s)e^{-\tau_1 s} + h(s)e^{-\tau_2 s} = 0
\]  

(1)

with:

\[
h(s) = \frac{C(s)G_0(s)}{1 + C(s)G_0(s)} ; \quad \tau_1 = \bar{\tau} ; \quad \tau_2 = \bar{\tau} + \Delta
\]
The geometric approach

The characteristic equation can be represented in the complex plane as an isosceles triangle.

The characteristic equation is equivalent to the following conditions:

1. The triangular inequality must hold for the triangle so that: $|h(j\omega)| \geq 1/2$;
2. It must satisfy the phase rule;
3. The sum of the internal angles of the isosceles triangle must be equal to $\pi$;

The method parametrizes the stability crossing curves in the delay plane $\tau_1, \tau_2$. 

The geometric approach

The characteristic equation can be represented in the complex plane as an isosceles triangle.
Stability crossing curves (SCC)

- Condition 1 (triangular inequality) $\Rightarrow$ By imposing $|h(j\omega)| \geq 1/2$ the *Frequency crossing set* $\Omega$ is obtained that is the union of a finite number of intervals of finite amplitude:

$$\Omega = \bigcup_{i=1}^{n} \Omega_i$$

- By imposing the other conditions it is obtained for $\omega \in \Omega$:

$$\tau_1^{u\pm}(\omega) = \frac{\angle h(j\omega) + 2u\pi}{\omega} \pm q(\omega) \quad ; \quad \tau_2^{v\pm}(\omega) = \frac{\angle h(j\omega) + (2v - 1)\pi}{\omega} \pm q(\omega)$$

$$q(\omega) = \arccos \left( \frac{1}{2|h(j\omega)|} \right)$$

where $u$ e $v$ are integers. The stability crossing curves $\mathcal{T}$ are obtained by drawing $\tau_1(\omega), \tau_2(\omega)$ for all $\omega \in \Omega$ and for all $u$ and $v$. 
For the considered network congestion control model $\overline{\tau} = T_1 + T_2$, $\tau_1 = \overline{\tau}$, $\tau_2 = \overline{\tau} + \Delta$ and:

$$h(s) = \frac{k}{s + k}$$

by making the change of variable $z = \frac{s}{k}$ (scaling of the closed-loop eigenvalues by $1/k$) we reduce the free parameters to two:

$$h(z) = \frac{1}{z + 1} ; h_1 = k\tau_1 ; h_2 = k\tau_2 \Rightarrow$$

$$1 - \frac{1}{1 + j\omega} e^{-j\omega h_1} + \frac{1}{1 + j\omega} e^{-j\omega h_2} = 0$$
By imposing $|h(j\omega)| \geq 1/2$ we obtain the frequency crossing set:

$$\Omega = (0, \sqrt{3}]$$

and by varying $\omega \in \Omega$, the sign and $u$ and $v$ in the integers we obtain the stability crossing curves $T$ parametrized as follows:

$$h_1^{u\pm}(\omega) = \frac{-\arctan \omega + 2u\pi \pm \arccos \left( \frac{\sqrt{1+\omega^2}}{2} \right)}{\omega}$$

$$h_2^{v\pm}(\omega) = \frac{-\arctan \omega + (2v - 1)\pi \mp \arccos \left( \frac{\sqrt{1+\omega^2}}{2} \right)}{\omega}$$
Stability crossing curves of the considered system

Problem

Find the maximum uncertainty \( \delta \) such that the system is asymptotically stable for all \( \Delta \in [-\delta, \delta] \). Thus we have to solve this problem:

\[
\delta = \min \min \min_{h_1^* \in \mathbb{R}^+, u, v, \tau_2^\pm \in \mathcal{T}} |h_2^{\pm} - h_1^*|
\]

This problem is equivalent to finding the minimum distance between a generic point of the positive bisector and a generic SCC. Thus it is sufficient to impose \( \frac{dh_1}{dh_2} = 1 \iff \frac{dh_1}{d\omega} = \frac{dh_2}{d\omega} \).
Stability crossing curves of the considered system

By considering the closest curves to the positive bisector we can restrict to the subset of stability crossing curves:

$$\overline{T} \subset T = T_{u,u} \cup T_{u,u+1}$$

For the positive curves \( v = u + 1 \), for the negative ones \( u = v \) (see figure).

By considering the positive curves \( v = u + 1 \) and imposing

\[
\frac{dh_1}{d\omega} = \frac{dh_2}{d\omega}
\]

it turns out:

\[
\arccos \left( \frac{\sqrt{\omega^2 + 1}}{2} \right) + \frac{\omega^2}{\sqrt{\omega^2 + 1} \sqrt{3 - \omega^2}} = \frac{\pi}{2}
\]
Robust Stability Analysis

By solving the equation for $\omega$ we obtain a unique solution in $\Omega$, $\overline{\omega} = 1.3483 \frac{\text{rad}}{s}$. Let us substitute $\overline{\omega}$ in $h_1(\omega)$ and $h_2(\omega)$, obtaining:

$$h_1(\overline{\omega}) = h_1 = 4.6601u - 0.2654$$

$$h_2(\overline{\omega}) = h_2 = 4.6601v - 3.4480$$

By subtracting the two equations ($v = u + 1$) we find that the tangents parallel to the positive bisector belonging to the curves $\mathcal{T}^+_{u,u+1}$ lie on the line:

$$h_2 = h_1 + 1.4775$$
Robust Stability Analysis

Proposition

A necessary and sufficient condition for the asymptotic stability of the system regardless the value of the nominal delay $\bar{\tau} = T_1 + T_2$ is that $|\Delta| < \frac{\alpha}{k}$

By recalling that $h_2 = h_1 + 1.4775$ and $h_1 = k\tau_1$, $h_2 = k\tau_2$ we obtain the condition:

$$k(\tau_2 - \tau_1) < 1.4775 \Rightarrow$$

$$\Delta < \frac{1.4775}{k}$$
Simulation set-up

- Matlab SIMULINK model
- Available bandwidth $b(t)$: step function $t = 1$ sec, final value $b = 100$ packets/sec
- Queue set point $w(t)$: step function at $t = 0$ sec, final value $w = 150$ packets.
- Nominal RTT: $\overline{\tau} = 1$ sec.
- Controller gain: $k = 4 \text{ sec}^{-1}$, thus the maximum delay uncertainty is $\delta \approx 0.37$ sec.
- Delay uncertainty $\Delta$: either 0, or $\delta/2$ or $\delta$
Simulation results

- **Queue length** $q(t)$: as expected, the performance degrades as $\Delta$ increases, providing an acceptable response for $\Delta = \frac{\delta}{2}$, whereas when $\Delta = \delta$ persistent oscillations occur.
Simulation results

- **Queue length q(t):** as expected, the performance degrades as $\Delta$ increases, providing an acceptable response for $\Delta = \delta/2$, whereas when $\Delta = \delta$ persistent oscillations occur.

- **Input Rate r(t):** The input rate is able to track the available bandwidth $b(t)$, but when $\Delta = \delta$ persistent oscillation occurs.
Conclusions

- We found a simple necessary and sufficient condition on the gain of the proportional controller $k$ in order to retain asymptotic stability regardless the value of the nominal delay $\tau = T_1 + T_2$ by employing a geometric approach.
- The maximum uncertainty allowed does not depend on the nominal delay $\tau$. This makes the controller effective even with large delays.
- The condition $|\Delta| < \alpha/k$ expresses a natural trade-off between the maximum delay mismatch $\delta$ and the proportional gain that can be used to tune the controller gain $k$.
- Congestion control algorithms that employ controllers made by a Smith predictor plus a proportional gain can be easily tuned in order to be robust to a bounded delay uncertainty.