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Controlling chaos via backstepping design

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In this Brief Report a method is presented to achieve both stabilization of chaotic motion to a steady state and tracking of any desired trajectory. The proposed approach is based on backstepping design and consists in a recursive procedure that interlaces the choice of a Lyapunov function with the design of feedback control. The main feature of this technique is that it gives the flexibility to build a control law by avoiding cancellations of useful nonlinearities, so that the goals of stabilization and tracking are achieved with a reduced control effort. A comparison with the differential geometric method clearly highlights the advantages of the proposed approach. [S1063-651X(97)12110-9]

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Many mechanical, chemical, or electronic systems can exhibit chaotic dynamics [1–3]. Since chaos is unpredictable and may lead to vibrations and fatigue failures in mechanical systems, its suppression is generally advantageous. Consequently, the control of chaos has received great interest in recent years [4–8]. The early studies have been devoted to the stabilization of periodic orbits embedded in a chaotic attractor via the Ott-Grebogi-York (OGY) technique [5]. However, since steady state solutions represent the most practical operation mode in many chaotic systems such as electronic oscillators [3] or laser systems [6], it is important to develop control technique to drive a strange attractor not only to a periodic orbit but also to a steady state. To solve this problem, the occasional proportional feedback (OPF) technique or conventional linear feedback methods have been proposed [7,8]. Unfortunately, since these methods give nonlinear closed-loop systems, the corresponding dynamic analysis is not simple and the computation of the feedback gains is trial and error. These drawbacks can be overcome by applying the differential geometric approach as shown in [9], where the idea is to algebraically transform a nonlinear system dynamics into a linear one so that linear control techniques can be utilized.

In this Brief Report a further contribution for controlling chaos is given. The proposed approach, based on backstepping design, consists in a recursive procedure that interlaces the choice of a Lyapunov function with the design of feedback control [10,11]. The key idea is to utilize the Lyapunov method by breaking the design problem for the full system into a sequence of design problems for lower-order (even scalar) systems. Thus, by exploiting the flexibility assured by lower-order and scalar systems, backstepping design can solve stabilization and tracking problems under conditions less restrictive than those encountered in other techniques. In particular the proposed approach can be applied to the following class of nonlinear strict-feedback systems [10]:

$$\dot{x}=f(x)+g(x)\xi_1,$$

$$\dot{\xi}_1=f_1(x,\xi_1)+g_1(x,\xi_1)\xi_2,$$

$$\vdots$$

$$\dot{\xi}_{k-1}=f_{k-1}(x,\xi_1,\ldots,\xi_{k-1})+g_{k-1}(x,\xi_1,\ldots,\xi_{k-1})\xi_k,$$

$$\dot{\xi}_k=f_k(x,\xi_1,\ldots,\xi_k)+g_k(x,\xi_1,\ldots,\xi_k)u, \tag{1}$$

where f and g are nonlinear functions with $x\in\mathcal{R}^n$, whereas $\xi_1,\xi_2,\ldots,\xi_k$ and u are scalars.

The nonlinearities f_i and g_i in the $\dot{\xi}_i$ equation ($i=1,...,k$) depend only on $x,\xi_1,\xi_2,\ldots,\xi_i$, i.e., on state variables that are "fed back." Backstepping design starts considering the variable ξ_1 as a "virtual control input" to stabilize the first equation. When ξ_1 has been designed, it goes on by considering the variable ξ_2 as the virtual control for the second equation, and so on. Therefore the design of the actual input $u(x)$ is systematically achieved in n steps [10,11].

It is worth noting that several chaotic systems, such as Rössler's chaotic system [12], Chua's circuit [3], the Lorenz system [13], and Rössler's hyperchaotic system [14], belong to the class of strict-feedback systems. However, in order to illustrate the capabilities and the advantages of the proposed approach, the attention is focused on the Lorenz system. This because it has turned out that Lorenz equations can model other physical systems and are of practical importance to

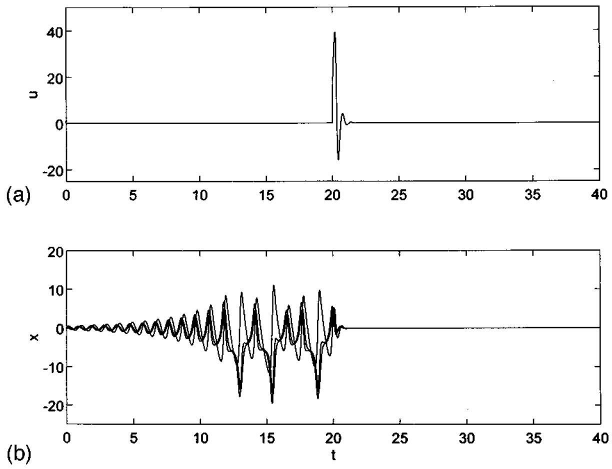

FIG. 1. Stabilization via backstepping design: (a) time wave form of the control u switched on at $t=20$; (b) time wave forms of x_1, x_2, and x_3.

implement circuits for secure communications [15]. The system considered herein is described by the following differential equations [1,9]:

$$\dot{x}=-px+py,$$

$$\dot{y}=-xz-y,$$

$$\dot{z}=xy-z-R, \tag{2}$$

where $R=R_0+u$ is the Rayleigh number, R_0 is the operation value, $p=10$ is the Prandtl number, and u is the control parameter. If $R_0=28$, the uncontrolled system (i.e., $u=0$) is chaotic and there are three unstable equilibrium points: $(C_0,C_0,-1)$, $(0,0,-R_0)$, and $(-C_0,-C_0,-1)$ where $C_0=\sqrt{R_0-1}$. It is worth noting that when the set point is the state $(C_0,C_0,-1)$ the OGY method is not applicable [9]. By translating the origin of system (2) in the set point $(C_0,C_0,-1)$, the system equations become

$$\dot{x}_1=-10x_1+10x_2,$$

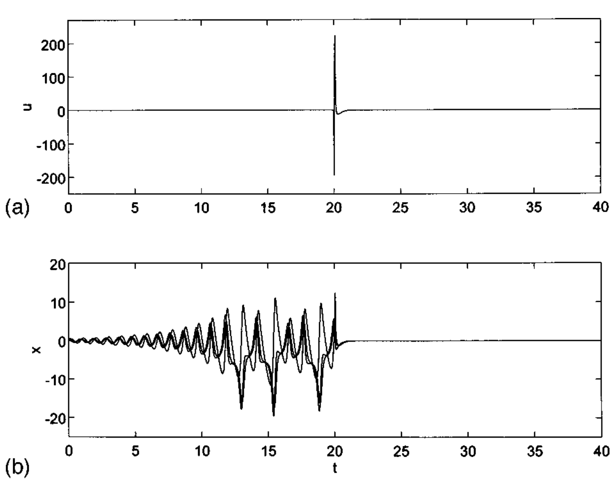

FIG. 2. Stabilization via differential geometric method: (a) time wave form of the control u switched on at $t=20$; (b) time wave forms of x_1, x_2, and x_3.

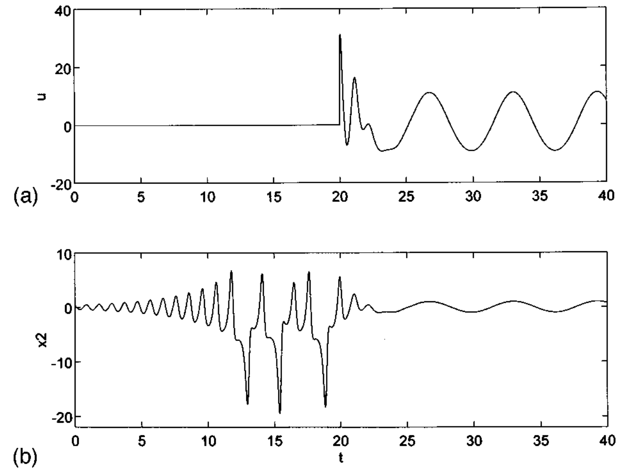

FIG. 3. Tracking of sin(t) via backstepping design: (a) time wave form of the control u switched on at $t=20$; (b) time wave form of the output x_2.

$$\dot{x}_2=x_1-x_2-(\sqrt{27}+x_1)x_3,$$

$$\dot{x}_3=\sqrt{27}(x_1+x_2)-x_3+x_1x_2-u. \tag{3}$$

The objective is to find a control law u for *stabilizing* the state of system (3) in the origin. Starting from the first equation, a *stabilizing function* $\alpha_1(x_1)$ has to be designed for the *virtual control* x_2 in order to make the derivative of $V_1(x_1)=x_1^2/2$,

$$\dot{V}_1=-10x_1^2+10x_1x_2,$$

negative definite when $x_2=\alpha_1(x_1)$. By choosing $\alpha_1(x_1)=0$ and by defining the *error variable* z_2:

$$z_2=x_2-\alpha_1(x_1) \tag{4}$$

the following (x_1,z_2) subsystem is obtained:

$$\dot{x}_1=-10x_1+10z_2,$$

$$\dot{z}_2=x_1-z_2-(\sqrt{27}+x_1)x_3,$$

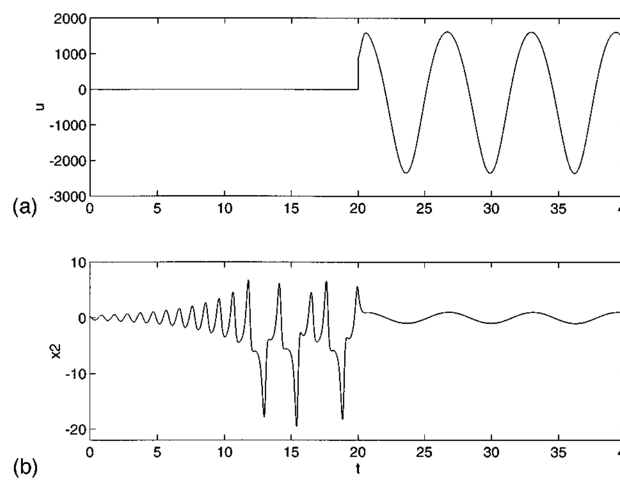

FIG. 4. Tracking of sin(t) via differential geometric method: (a) time wave form of the control u switched on at $t=20$; (b) time wave form of the output x_2.

for which a candidate Lyapunov function is $V_2(x_1,z_2)=V_1(x_1)+\frac{1}{2}z_2^2$. Since its time derivative

$$\dot{V}_2=-10x_1^2+z_2[11x_1-z_2-(\sqrt{27}+x_1)x_3]$$

becomes negative definite by choosing the virtual control x_3 as

$$x_3=\alpha_2(x_1,z_2)=\frac{11x_1}{\sqrt{27}+x_1}$$

the deviation of x_3 from the stabilizing function α_2,

$$z_3=x_3-\frac{11x_1}{\sqrt{27}+x_1}, \tag{5}$$

gives the following system in the (x_1,z_2,z_3) coordinates:

$$\dot{x}_1=-10x_1+10z_2,$$

$$\dot{z}_2=x_1-z_2-(\sqrt{27}+x_1)(z_3+\alpha_2),$$

$$\dot{z}_3=[\sqrt{27}(x_1+z_2)-z_3-\alpha_2+x_1z_2-u]-\left[\frac{11\sqrt{27}}{(\sqrt{27}+x_1)^2}(-10x_1+10z_2)\right].$$

By iterating the previous steps, the derivative of $V_3(x_1,z_2,z_3)=V_2+\frac{1}{2}z_3^2$,

$$\dot{V}_3=-10x_1^2-z_2^2+z_3\left[\sqrt{27}x_1-z_3-\frac{11x_1}{\sqrt{27}+x_1}-u-\frac{11\sqrt{27}(-10x_1+10x_2)}{(\sqrt{27}+x_1)^2}\right],$$

becomes negative definite by choosing the input

$$u=\sqrt{27}x_1-\frac{11x_1}{\sqrt{27}+x_1}-\frac{11\sqrt{27}(-10x_1+10x_2)}{(\sqrt{27}+x_1)^2}, \tag{6}$$

which proves that in the (x_1,z_2,z_3) coordinates the origin has been stabilized. In view of Eqs. (4) and (5) the origin in the (x_1,x_2,x_3) coordinates has the same properties. It can be concluded that Eq. (6) represents the control law for stabilizing system (2) in $(C_0,C_0,-1)$.

Now, the goal is to find a control law u such that a scalar output tracks any desired trajectory $r(t)$, including stable or unstable limit cycles as well as chaotic trajectories. Let $y=x_2$ be the output and let z_2 be the deviation of x_2 from the target, i.e., $z_2=x_2-r(t)$. Given $V_2=z_2^2/2$, its time derivative,

$$\dot{V}_2=z_2[x_1-z_2-r(t)-(\sqrt{27}+x_1)x_3-\dot{r}(t)],$$

becomes negative by choosing the virtual control x_3 as

$$x_3=\alpha_2=\frac{x_1-r-\dot{r}}{\sqrt{27}+x_1}.$$

Again, given $V_3=V_2+z_3^2/2$, where $z_3=x_3-\alpha_2$ is the deviation of the virtual control from the stabilizing function, the time derivative

$$\dot{V}_3=-z_2^2-z_3[z_2\sqrt{27}+z_2x_1-\sqrt{27}(x_1+x_2)+z_3+\alpha_2-x_1x_2+u+\dot{\alpha}_2]$$

is negative by choosing the input

$$u=\frac{(10x_1-10x_2)(\sqrt{27}+r+\dot{r})}{(\sqrt{27}+x_1)^2}+\frac{2\dot{r}+\ddot{r}-x_1+r}{\sqrt{27}+x_1}+r(\sqrt{27}+x_1)+\sqrt{27}x_1, \tag{7}$$

which assures that x_2 tracks the reference signal r.

Now a comparison with the differential geometric method [9] is carried out. As far as the authors are aware, this technique was the only one able to control chaos in a systematic way. To illustrate the advantages of the proposed approach, numerical simulations concerning stabilization by using the backstepping design are reported in Fig. 1, whereas the results obtained via the differential geometric method are shown in Fig. 2. With reference to tracking, the results are reported in Figs. 3 and 4, respectively. All these figures highlight the fact that backstepping design is more efficient because it requires less control effort than the differential geometric method. The reason is that backstepping pursues the goals of stabilization and tracking rather than that of linearization.

In conclusion, an approach to control chaos based on backstepping design has been presented. The advantages of the suggested technique can be summarized as follows: (1) it is a systematic procedure for controlling chaos; (2) it can be applied to several chaotic systems; (3) both stabilization and tracking can be achieved even if the target is outside the strange attractor; and (4) it requires less control effort in comparison with the differential geometric method.

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