Robust Stability Analysis of a Class of Smith Predictor-based Congestion Control Algorithms for Computer Networks

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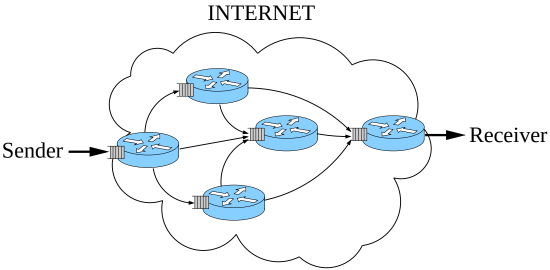
Outline

Introduction



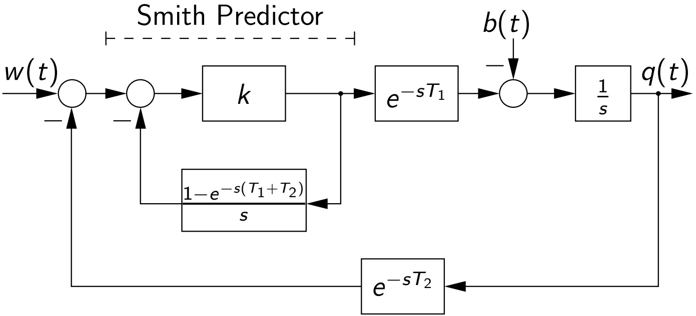
- The Internet is a relevant example of time-delay system due to propagation of information over communication links
- When packets arrive at a rate above the capacity of the output link the router queue builds and congestion arises
- TCP Congestion control is a building block of the Internet designed to avoid congestion and preserve network stability

TCP Congestion Control Models

The literature dealing with congestion control algorithms modeling is very broad:

- **Stationary models**: predict the average throughput based on average measures such as round trip time, packet loss ratio
  - SQRT Formula (*Mathis et al., 1997*)
  - PFTK formula (*Padhye et al., 1998*)
- **Fluid Models**: based on differential equations
  - *Hollot et al. (2001)*: non-linear differential equations of congestion window $w(t)$ and queue length $q(t)$
  - *Mascolo (1999)*: linear model comprising two time delays, an integrator and a Smith Predictor.
- **Hybrid Models**: time continuous differential equations coupled with discrete dynamics (packet loss events, etc)
  - *Hespana et al. (2001)*: describes all the phases of TCP congestion control (slow start, cong. avoid., fast retx)

Smith-predictor based Congestion Control
(S. Mascolo, 1999)



$w(t)$: Set-point (congestion window)
$q(t)$: Queue length
T_1, T_2: source-destination and destination-source delays

k: Controller gain
$1/s$: bottleneck link queue model
$b(t)$: bottleneck available bandwidth

Why using a Smith Predictor plus proportional controller?

- If the delay $T_1 + T_2$ (RTT, round trip time) is exactly known, the closed loop dynamics is that of a first order system (no overshoots, system is always stable)
- A unique parameter to tune having a direct influence on the step response of the system
- It models the TCP congestion control and its variants by proper input shaping of the congestion window $w(t)$
- A rate-based congestion control algorithm has been designed and implemented (Grieco and Mascolo, 2004)

Focus of the paper

Smith-predictor is known to be sensitive to delay model uncertainties. What's the effect of a measurement error in the delay on the stability of the system?

Stability crossing curves for systems with two delays
(Gu, Niculescu, Chen, 2005)

Let us consider:

- LTI SISO system $G(s) = G_0(s)e^{-\tau s}$, $G_0(s)$ delay free plant
- Smith predictor controller $C(s)$, nominal delay $\overline{\tau} = T_1 + T_2$, Δ delay uncertainty

Characteristic equation:

$$\boxed{1 - h(s)e^{-\tau_1 s} + h(s)e^{-\tau_2 s} = 0} \qquad (1)$$

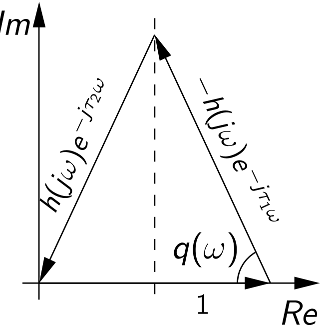
with:

$$h(s) = \frac{C(s)G_0(s)}{1 + C(s)G_0(s)} \; ; \;\; \tau_1 = \overline{\tau} \;\; ; \;\; \tau_2 = \overline{\tau} + \Delta$$

The geometric approach

Geometric approach

The characteristic equation can be represented in the complex plane as an isoceles triangle



The characteristic equation is equivalent to the following conditions

1. The triangular inequality must hold for the triangle so that: $|h(j\omega)| \geq 1/2$;
2. It must satisfy the phase rule;
3. The sum of the internal angles of the isosceles triangle must be equal to π;

The method parametrizes the stability crossing curves in the delay plane τ_1, τ_2

Stability crossing curves (SCC)

- Condition 1 (triangular inequality) $\Rightarrow$ By imposing $|h(j\omega)| \geq 1/2$ the *Frequency crossing set* Ω is obtained that is the union of a finite number of intervals of finite amplitude:

$$\Omega = \bigcup_{i=1}^{n} \Omega_i$$

- By imposing the other conditions it is obtained for $\omega \in \Omega$:

$$\tau_1^{u\pm}(\omega) = \frac{\angle h(j\omega) + 2u\pi \pm q(\omega)}{\omega}\ ;\ \tau_2^{v\pm}(\omega) = \frac{\angle h(j\omega) + (2v-1)\pi \mp q(\omega)}{\omega}$$

$$q(\omega) = \arccos\left(\frac{1}{2|h(j\omega)|}\right)$$

where u e v are integers. The stability crossing curves $\mathcal{T}$ are obtained by drawing $\tau_1(\omega), \tau_2(\omega)$ for all $\omega \in \Omega$ and for all u and v.

SCC of the computer network congestion control model

For the considered network congestion control model $\overline{\tau} = T_1 + T_2$, $\tau_1 = \bar{\tau}$, $\tau_2 = \overline{\tau} + \Delta$ and:

$$h(s) = \frac{k}{s+k}$$

by making the change of variable $z = \frac{s}{k}$ (scaling of the closed-loop eigenvalues by $1/k$) we reduce the free parameters to two:

$$h(z) = \frac{1}{z+1}\,;\ h_1 = k\tau_1\,;\ h_2 = k\tau_2 \ \Rightarrow$$

$$\boxed{1 - \frac{1}{1+j\omega}e^{-j\omega h_1} + \frac{1}{1+j\omega}e^{-j\omega h_2} = 0}$$

SCC of the computer network congestion control model

By imposing $|h(j\omega)| \geq 1/2$ we obtain the *frequency crossing set:*

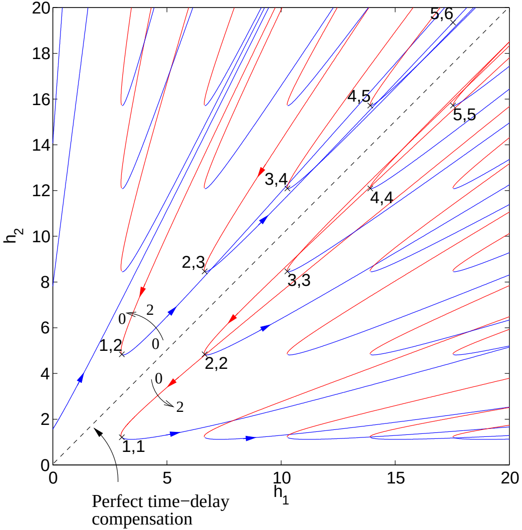
$$\Omega = (0, \sqrt{3}]$$

and by varying $\omega \in \Omega$, the sign and u and v in the integers we obtain the stability crossing curves $\mathcal{T}$ parametrized as follows:

$$h_1^{u\pm}(\omega) = \frac{-\arctan\omega + 2u\pi \pm \arccos\left(\frac{\sqrt{1+\omega^2}}{2}\right)}{\omega}$$

$$h_2^{v\pm}(\omega) = \frac{-\arctan\omega + (2v-1)\pi \mp \arccos\left(\frac{\sqrt{1+\omega^2}}{2}\right)}{\omega}$$

Stability crossing curves of the considered system



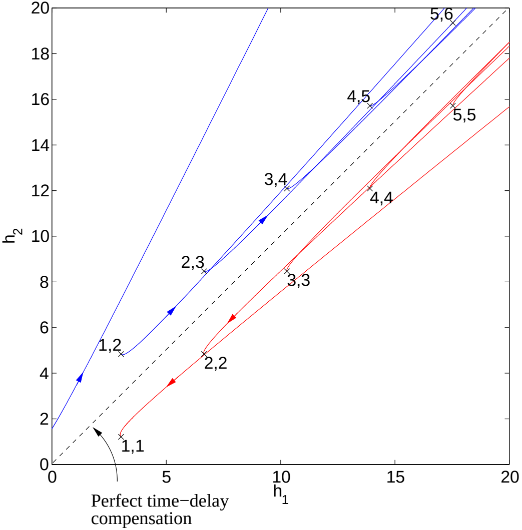
Problem

Find the maximum uncertainty δ such that the system is asymptotically stable for all $\Delta \in [-\delta, \delta]$. Thus we have to solve this problem:

$$\delta = \min_{h_1^* \in \mathbb{R}_+} \min_{u,v} \min_{\tau_2^{v\pm} \in \mathcal{T}} |h_2^{v\pm} - h_1^*|$$

This problem is equivalent to finding the minimum distance between a generic point of the positive bisector and a generic SCC. Thus it is sufficient to impose $\frac{dh_1}{dh_2} = 1 \Leftrightarrow \frac{dh_1}{d\omega} = \frac{dh_2}{d\omega}$

Stability crossing curves of the considered system



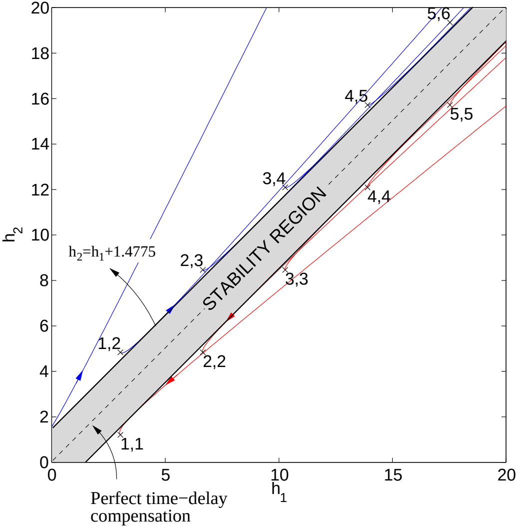
By considering the closest curves to the positive bisector we can restrict to the subset of stability crossing curves:

$$\overline{\mathcal{T}} \subset \mathcal{T} = \mathcal{T}^{-}_{u,u} \cup \mathcal{T}^{+}_{u,u+1}$$

For the *positive* curves $v = u + 1$, for the *negative* ones $u = v$ (see figure).
By considering the *positive* curves ($v = u + 1$) and imposing $\frac{dh_1}{d\omega} = \frac{dh_2}{d\omega}$ it turns out:

$$\arccos\left(\frac{\sqrt{\omega^2+1}}{2}\right) + \frac{\omega^2}{\sqrt{\omega^2+1}\sqrt{3-\omega^2}} = \frac{\pi}{2}$$

Robust Stability Analysis



By solving the equation for ω we obtain a unique solution in Ω, $\overline{\omega} = 1.3483 \frac{rad}{s}$. Let us substitute $\overline{\omega}$ in $h_1(\omega)$ and $h_2(\omega)$, obtaining:

$$\begin{aligned} h_1(\overline{\omega}) &= h_1 = 4.6601u - 0.2654 \\ h_2(\overline{\omega}) &= h_2 = 4.6601v - 3.4480 \end{aligned}$$

By subtracting the two equations ($v = u + 1$) we find that the tangents parallel to the positive bisector belonging to the curves $\mathcal{T}^+_{u,u+1}$ lie on the line:

$$h_2 = h_1 + 1.4775$$

Robust Stability Analysis

Proposition

A necessary and sufficient condition for the asymptotic stability of the system regardless the value of the nominal delay $\overline{\tau} = T_1 + T_2$ is that $|\Delta| < \frac{\alpha}{k}$

By recalling that $h_2 = h_1 + 1.4775$ and $h_1 = k\tau_1$, $h_2 = k\tau_2$ we obtain the condition:

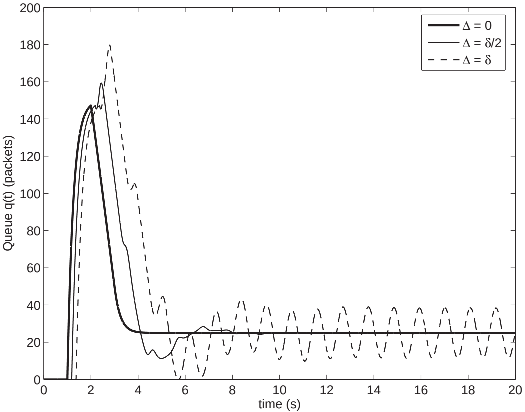
$$k(\tau_2 - \tau_1) < 1.4775 \Rightarrow$$

$$\boxed{\Delta < \frac{1.4775}{k}}$$

Simulation set-up

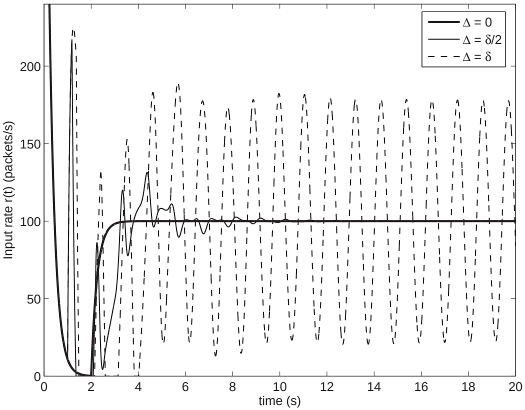
- Matlab SIMULINK model
- *Available bandwidth* $b(t)$: step function $t = 1\,\text{sec}$, final value $b = 100\,\text{packets}/\text{sec}$
- *Queue set point* $w(t)$: step function at $t = 0\,\text{sec}$, final value $w = 150\,\text{packets}$.
- *NominalRTT*: $\overline{\tau} = 1\,\text{sec}$.
- *Controller gain*: $k = 4\,\text{sec}^{-1}$, thus the maximum delay uncertainty is $\delta \cong 0.37\,\text{sec}$.
- *Delay uncertainty* Δ: either 0, or $\delta/2$ or δ

Simulation results



- *Queue length* $q(t)$: as expected, the performance degrades as Δ increases, providing an acceptable resposnse for $\Delta = \delta/2$, whereas when $\Delta = \delta$ presistent oscillations occur.

Simulation results



- *Queue length* $q(t)$: as expected, the performance degrades as Δ increases, providing an acceptable resposnse for $\Delta = \delta/2$, whereas when $\Delta = \delta$ presistent oscillations occur.
- *Input Rate* $r(t)$: The input rate is able to track the available bandwidth $b(t)$, but when $\Delta = \delta$ persistent oscillation occurs.

Conclusions

- We found a simple necessary and sufficient condition on the gain of the proportional controller k in order to retain asymptotic stability regardless the value of the nominal delay $\overline{\tau} = T_1 + T_2$ by employing a geometric approach.
- The maximum uncertainty allowed does not depend on the nominal delay $\overline{\tau}$. This makes the controller effective even with large delays.
- The condition $|\Delta| < \alpha/k$ expresses a natural trade-off between the maximum delay mismatch δ and the proportional gain that can be used to tune the controller gain k.
- Congestion control algorithms that employ controllers made by a Smith predictor plus a proportional gain can be easily tuned in order to be robust to a bounded delay uncertainty.