The Trade-off between Robustness and Disturbance Rejection for Congestion Control Algorithms based on a Modified Smith-Predictor

Luca De Cicco, Saverio Mascolo, Silviu-Iulian Niculescu

Abstract— **Congestion control is a cornerstone component of the Internet. The plant dynamics can be modelled by means of an integrator, modelling a bottleneck queue, a time delay, modelling the propagation of the information from a source to a destination along with queuing, and a load disturbance, which models the time-varying available bandwidth. It has been shown that a Smith predictor plus a proportional gain is an effective controller, even though, it is not able to reject the load disturbance. To overcome this issue, which is particularly relevant in the case of multimedia delay-sensitive traffic, we consider the modified Smith predictor proposed by Matausek and Micic as a candidate for the design of a congestion control algorithm. By taking a geometric approach, we quantify the trade-off between disturbance rejection property of the modified Smith predictor and the achievable stability robustness with respect to delay uncertainty, which in data networks is due to queuing. Finally, we propose some guidelines to tune the additional parameter introduced by the modified Smith predictor.**

Index Terms— **Time-delay systems, Robust stability, congestion control, Smith predictor**

I. INTRODUCTION

Integrative plants with time-delays represent an important class of systems which can be frequently encountered in several domains such as industrial plants, communication networks, and supply chain management [1], [2]. The design of an effective controller for such systems is particularly challenging, since the plant is not asymptotically stable and it contains a time-delay which may have adverse effects on the stability of the closed-loop system [3]. Among the approaches proposed in the literature, the Smith predictor has been widely studied and applied due to its effectiveness in compensating the time-delay [4]. However, it is well-known that the Smith predictor is affected by two main drawbacks, namely 1) it is not able to reject a load disturbance for processes containing integrator dynamics [5] and 2) it requires the exact knowledge of both the plant model and the time delay to provide effective delay compensation [6]. Regarding the first issue, several modifications to the Smith predictor scheme were proposed. In the case of processes in which disturbance is not measurable, the idea is to design a *disturbance estimator* such that the set-point response can be decoupled by the disturbance response [5], [7]. Recently a two-degree of freedom modified Smith-predictor (MSP) controller has been proposed and its robustness properties have been studied [8]. In the case of measurable delays, feed-forward techniques may be used with the classic Smith predictor controller [9].

In this paper we consider computer networks, whose most relevant example is the Internet, which can be modelled as time-delay systems since the information, which is sent in the form of packets, propagates from a source through a number of communication links and queues before reaching its destination [1]. A fundamental building block of the Internet is the end-to-end congestion control algorithm that was designed and implemented in the 80s in the Transmission Control Protocol (TCP), and which has been able, since then, to avoid network congestion and provide network stability.

A large body of literature is available on the mathematical modelling and design of congestion control algorithms employing different approaches, namely fluid models, hybrid systems, $\mathcal{H}^\infty$ design [1], [10], [11], [12]. The plant dynamics can be modelled by means of an integrator, modelling a bottleneck queue, a time delay, modelling the propagation of the information from a source to a destination along with queuing, and a load disturbance, which models the time-varying available bandwidth [1]. In [1] it has been shown that a Smith-predictor plus a proportional gain is an effective controller, even though it is not able to reject the load disturbance (i.e. the available bandwidth) and the queue matches the set-point with a steady-state error.

In this paper we explore the possibility of using the modified Smith predictor controller described in [7] to obtain a zero steady-state error. We employ the approach taken in [7] since the introduced disturbance estimator block has only one parameter to tune having a clear physical interpretation. Even though it is not the main goal of a congestion control algorithm, having zero steady state error is an interesting feature, since such a controller could provide full link utilization with small queues. Reducing the queuing delay is particularly important for delay-sensitive traffic, such as the one generated by video conference applications, which require packet delivery delay to be kept low.

The main contribution of this paper is the quantification of the trade-off between the disturbance rejection property of the modified Smith predictor [7] and the achievable stability robustness with respect to delay uncertainty, which in data networks is due to queuing. From the design point of view, this also allows us to give some guidelines for the tuning of the additional parameter introduced by the disturbance estimator. It is worth noting that even though comparison between Smith predictor control schemes was done in the past, the robustness issue *wrt* delay mismatches was not

Luca De Cicco and Saverio Mascolo are with the Dipartimento di Ingegneria Elettrica e dell'Informazione, Politecnico di Bari, Via Orabona 4, e-mails: l.decicco@poliba.it, mascolo@poliba.it

Silviu-Iulian Niculescu is with Supelec, Laboratoire des Signaux et Systemes, 3 rue Joliot-Curie 91192 Gif-sur-Yvette cedex, France Silviu.Niculescu@lss.supelec.fr

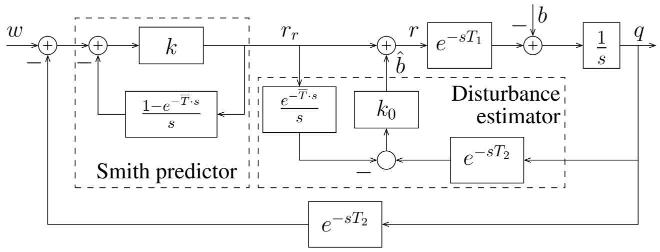


Fig. 1: Block diagram of the considered congestion control algorithm model

sufficiently explored in comparing the corresponding control schemes and, to the best of the authors knowledge, our contribution opens interesting perspectives in this direction.

II. The Congestion Control Algorithm

A network connection is made by a set of communication links, through which the information is sent, and a set of routers, where the information is temporarily stored in a queue before being forwarded to the next link. Congestion occurs when the total incoming rate at a router exceeds the capacity of its output link and, consequently, its queue builds up until the point packets start to be dropped.

Fig. 1 shows the block diagram of the considered congestion control algorithm model, that is made of the following blocks: 1) *two time delays*, T_1 which models the propagation delay of a packet from the source to the bottleneck queue, and T_2 which models the delay for propagation from the queue to the destination and back to the sender; $T_1+T_2 = T$ is the round-trip time of the connection; 2) an *integrator* $1/s$, modelling the bottleneck queue, which is filled (or drained) by the rate mismatch between the input rate $r(t-T_1)$ and the bottleneck available bandwidth $b(t)$; 3) a *Smith predictor*, using a proportional controller with gain k, which computes the control action $r_r(t)$; the nominal delay used in the Smith predictor is $\overline{T}$ that is an estimate of the actual round-trip time T; 4) a *disturbance estimator*, as proposed in [7], which produces an estimate $\hat{b}(t)$ of the disturbance $b(t)$.

With respect to the model presented in [1], the one shown in Fig. 1 includes the disturbance estimator proposed in [7], which adds a second parameter to tune, i.e. k_0. In [1] it has been proved that, when the disturbance estimator is not used ($k_0 = 0$), the Smith predictor models the TCP congestion control and the signal $w(t)$ models the congestion window (*cwnd*) or the advertisement window (*awnd*) that is used by the TCP to limit the outstanding packets [13], [1]. Moreover, it is well-known that, when $k_0 = 0$, the general system shown in Fig. 1 is not internally stable due to integral mode. However, in [14] it has been shown that the fact that the system is not internally stable does not pose any implementation issues in the case of the congestion control of computer network. Thus, the only drawback is that, when $k_0 = 0$, a step disturbance cannot be rejected at steady state.

III. Review of the geometric approach

In this Section we provide a brief review of the geometrical approach [15] that will be employed to analyze the robust stability of the system shown in Fig. 1. Let us consider an LTI system with two independent time delays τ_1 and τ_2:

$$\dot{x}(t) = A_0 x(t) + A_1 x(t-\tau_1) + A_2 x(t-\tau_2). \tag{1}$$

The stability of (1) can be studied by looking at the associated characteristic function:

$$p(s;\tau_1,\tau_2) = p_0(s) + p_1(s)e^{-s\tau_1} + p_2(s)e^{-s\tau_2}.$$

We aim at finding the couples $(\tau_1,\tau_2) \in \mathbb{R}_+^2$ such that the solutions of the characteristic equation $p(s;\tau_1,\tau_2) = 0$ are on the imaginary axis. By excluding some trivial cases (see [15]) we can refer to the following equation:

$$a(s;\tau_1,\tau_2) = 1 + a_1(s)e^{-s\tau_1} + a_2(s)e^{-s\tau_2} = 0. \tag{2}$$

The geometrical approach stems from the observation that the three terms of (2) can be considered as vectors in the complex plane. The equality $a(s;\tau_1,\tau_2) = 0$ can be represented in the complex plane via a triangle.

Therefore, (2) is equivalent to the following conditions:

1) The triangular inequality must hold for the triangle:

$$|a_1(j\omega)| + |a_2(j\omega)| \geq 1, \tag{3}$$

$$-1 \leq |a_1(j\omega)| - |a_2(j\omega)| \leq 1. \tag{4}$$

2) Equation (2) must satisfy the phase rule;
3) The sum of the internal angles of the triangle must be equal to π;

The solution of (3) and (4) does not depend on the time delays τ_1 and τ_2 and forms the *frequency crossing set* Ω. In [15] it has been shown that Ω is the union of a finite number of intervals of finite length $\Omega_1, \Omega_2, \ldots, \Omega_N$. For any $\omega > 0$ belonging to Ω there exists at least a pair (τ_1,τ_2) in the parameters space such that the system has at least one imaginary pole. By employing the conditions 2 and 3, it is possible to parametrize all the couples $(\tau_1,\tau_2) \in \mathbb{R}_+^2$ satisfying $a(j\omega;\tau_1,\tau_2) = 0$ using the following equations:

$$\tau_1^{u\pm} = \frac{\angle a_1(j\omega) + (2u-1)\pi \pm \theta_1}{\omega} \geq 0, \tag{5}$$

$$\tau_2^{v\pm} = \frac{\angle a_2(j\omega) + (2v-1)\pi \mp \theta_2}{\omega} \geq 0, \tag{6}$$

where $\omega \in \Omega$, u and v are integers such that $\tau_1^{u\pm}$, $\tau_2^{v\pm}$ are positive, and θ_1, θ_2 are the internal angles of the triangle given by:

$$\theta_1(\omega) = \arccos\left(\frac{1+|a_1(j\omega)|^2 - |a_2(j\omega)|^2}{2|a_1(j\omega)|}\right),$$

$$\theta_2(\omega) = \arccos\left(\frac{1+|a_2(j\omega)|^2 - |a_1(j\omega)|^2}{2|a_2(j\omega)|}\right).$$

To better understand the meaning of (5) and (6) let us fix $u = \overline{u}$ and $v = \overline{v}$: when ω varies in Ω_i and (5) and (6) are evaluated for both positive and negative signs, two curves in the parameter space (τ_1,τ_2) are obtained. We denote these curves with $\mathscr{T}_{\overline{u},\overline{v}}^{i+}$ and $\mathscr{T}_{\overline{u},\overline{v}}^{i-}$ respectively. In order to find out how each $\mathscr{T}_{u,v}^{i-}$ is connected to $\mathscr{T}_{u,v}^{i+}$ at the ends of the interval Ω_i, we observe that the end points of the intervals, denoted with ω_i^l and ω_i^r, must satisfy only one of the three following equations that we have to solve for (3) and (4) to hold:

$$|a_1(j\omega)| + |a_2(j\omega)| = 1, \tag{7}$$

$$|a_1(j\omega)| - |a_2(j\omega)| = 1, \tag{8}$$

$$|a_2(j\omega)| - |a_1(j\omega)| = 1. \tag{9}$$

It is possible to classify all these end points ω_i^l and ω_i^r in one of the following categories: 1) *Type 1*. Equation (8) is satisfied; in this case it can be shown that $\theta_1 = 0$ and $\theta_2 = \pi$ so that $\mathscr{T}_{u,v}^{i+}$ is connected to $\mathscr{T}_{u,v-1}^{i-}$ at this end; 2) *Type 2*. Equation (9) is satisfied; in this case $\theta_1 = \pi$ and $\theta_2 = 0$ and $\mathscr{T}_{u,v}^{i+}$ is connected to $\mathscr{T}_{u+1,v}^{i-}$ at this end; 3) *Type 3*. Equation (7) is satisfied; in this case it turns out $\theta_1 = \theta_2 = 0$ and $\mathscr{T}_{u,v}^{i+}$ is connected to $\mathscr{T}_{u,v}^{i-}$ at this end; 4) *Type 0*. $\omega_i^l = 0$; in this case as $\omega \to 0$ both $\mathscr{T}_{u,v}^{i+}$ and $\mathscr{T}_{u,v}^{i-}$ approach to ∞.

We define the *stability crossing curves* $\mathscr{T}$ in the τ_1, τ_2 plane as the union of all the curves $\mathscr{T}_{u,v}^{i}$ for $i \in \{1, \ldots, N\}$, and $u \in \mathbb{N}$, $v \in \mathbb{N}$. All the couples $(\tau_1, \tau_2) \in \mathscr{T}$ can be associated to a couple of poles on the imaginary axis which we call *crossing point*. Finally, when a stability crossing curve is crossed in the τ_1, τ_2 plane, two poles cross the imaginary axis on the complex plane [15].

IV. Robustness Analysis

Let us consider Fig. 1 which shows the block diagram of the considered system: k is the proportional controller gain, $\overline{\tau}$ and τ are respectively the nominal and the actual RTT, and k_0 is the gain of the disturbance estimator proposed in [7].

It is easy to show that the characteristic equation associated to the system shown in Fig. 1 is:

$$a_0(s; k, k_0) = 1 - \frac{ke^{-\overline{\tau}s}}{(s+k)} + \frac{(k_0+k)s + k_0 k}{s(s+k)} e^{-\tau s} = 0. \tag{10}$$

We are interested in characterizing the stability of the system when $\overline{\tau}$, τ, k, and k_0 vary in $\mathbb{R}_+$. By making the change of variable $z = s/k$, and by denoting $\alpha = k_0/k$, we obtain:

$$a_0(z; \alpha) = 1 - \frac{1}{z+1} e^{-h_1 z} + \frac{(\alpha+1)z + \alpha}{z(z+1)} e^{-h_2 z} = 0. \tag{11}$$

In (11) we have made the change of variable $h_1 = k\overline{\tau}$ and $h_2 = k\tau$, which reduces the free parameters from four to three. It is worth noting that, when $\alpha = 0$, the disturbance estimator is not effective and the standard Smith predictor is obtained.

In the following we shall consider separately the case $\alpha = 0$ (Section IV-A) and the case $\alpha > 0$ (Section IV-B) which corresponds to the modified Smith predictor (MSP) proposed in [7]. The method for finding a delay-dependent asymptotic stability condition for the closed loop system shown in Fig. 1 is first developed in the case of $\alpha = 0$ and then it is applied in the case of $\alpha > 0$.

A. The standard Smith predictor ($\alpha = 0$)

When $\alpha = 0$ (11), (5), and (6) turn out to be respectively:

$$a(z) = 1 - \frac{1}{z+1} e^{-h_1 z} + \frac{1}{z+1} e^{-h_2 z} = 0, \tag{12}$$

$$h_1^{u\pm}(\omega) = \frac{-\arctan\omega + 2u\pi \pm \arccos\left(\frac{\sqrt{1+\omega^2}}{2}\right)}{\omega}, \tag{13}$$

$$h_2^{v\pm}(\omega) = \frac{-\arctan\omega + (2v-1)\pi \mp \arccos\left(\frac{\sqrt{1+\omega^2}}{2}\right)}{\omega}. \tag{14}$$

By solving (3) and (4), the frequency crossing set $\Omega = (0, \sqrt{3}]$ is obtained. It can be shown that the stability crossing curves are a family of open ended curves with both ends approaching ∞ when $\omega \to 0$ [14].

The following delay-independent necessary and sufficient condition for the asymptotic stability of (12) can be established:

Proposition 1: A necessary and sufficient condition for the asymptotic stability of the system independent of the value of the nominal delay $\overline{\tau}$ is:

$$|\Delta| < \frac{a}{k}, \tag{15}$$

where $\Delta = \tau - \overline{\tau}$ is the delay uncertainty, $a \simeq 1.4775$ and k is the proportional gain of the controller.

Proof: The proof is given in [14]. ■

Nevertheless, when an estimate of the nominal delay $\overline{\tau}$ is known, the delay-independent condition (15) could be conservative. A delay-dependent tuning rule that is often considered in the literature is $k = \gamma/\overline{\tau}$ ($\gamma > 0$), i.e. the gain is inversely proportional to the nominal value of the delay: in [16] a value of $\gamma = 3$ is proposed, whereas in [17] a more conservative tuning rule is considered, i.e. $\gamma = 1$.

With the purpose of quantifying the robustness of the tuning rule $k = \gamma/\overline{\tau}$, we compute $\delta(\gamma)$ which we define as the maximum value of $k\Delta$ such that (12) has all its root in the left half complex plane. With the purpose of finding $\delta(\gamma)$, we fix γ and, by considering (13), we solve

$$h_1^{u\pm}(\omega) = \gamma \tag{16}$$

for $u \in \mathbb{N}$. For a fixed value of γ, (16) admits at least one solution for $u \in U \subset \mathbb{N}$, and it exists $\overline{u} \in \mathbb{N}$ such that for $u > \overline{u}$ (16) does not admit solutions, i.e. U in not equal to $\mathbb{N}$. Moreover, it is possible to show that for a fixed u, $h_1^{u+}(\omega) = \gamma$ and $h_1^{u-}(\omega) = \gamma$ admit at most two solutions each, that we denote with $\omega_A^{u+}, \omega_B^{u+}$ and $\omega_A^{u-}, \omega_B^{u-}$ respectively. Hence, we conclude that (16) has a finite number of solutions. Then, if we denote with χ_γ the set of the frequencies that solves (16), we can find $\delta(\gamma)$ as follows:

$$\delta(\gamma) = \min_{\omega \in \chi_\gamma, u \in U, v \in V} |h_2^{v\pm}(\omega) - h_1^{u\pm}(\omega)|. \tag{17}$$

Fig. 2 shows $\delta(\gamma)$ and the corresponding values in the cases of the tuning rules proposed in [17] and [16]. The figure shows that for increasing values of γ, $\delta(\gamma)$ increases

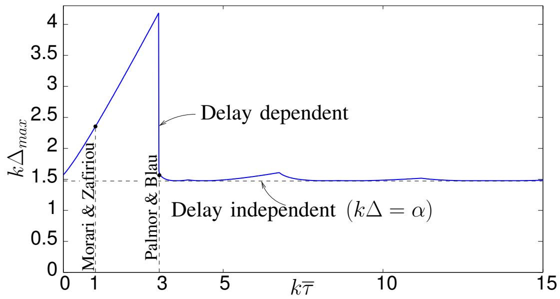


Fig. 2: Maximum delay uncertainty function of $k\overline{\tau} = \gamma$ in the case of the standard Smith predictor ($\alpha = 0$)

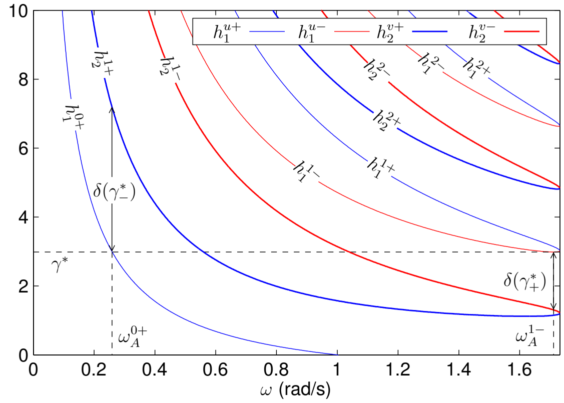


Fig. 3: $h_1^{u\pm}(\omega)$ and $h_2^{v\pm}(\omega)$ for $\omega \in \Omega$

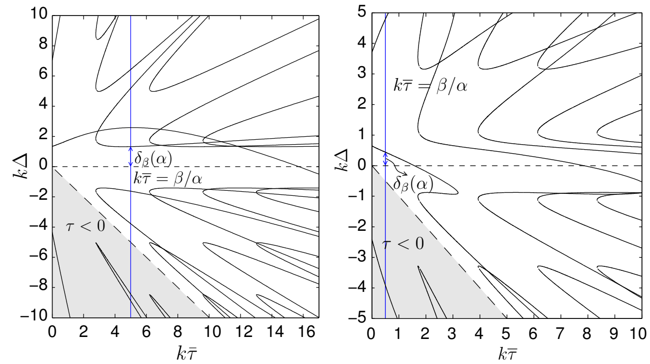
until a maximum is obtained for $\gamma = \gamma^*$ and then it abruptly decreases for $\gamma > \gamma^*$. To give an explanation of such phenomenon, Fig. 3 shows $h_1^{u\pm}(\omega)$ for $u = 0, 1, 2$ and $h_2^{v\pm}(\omega)$ for $v = 1, 2, 3$. By looking at Fig. 3, (16) corresponds in finding all the intersections of the line parallel to the the ω axis and ordinate γ with the family of curves $h_1^{u\pm}(\omega)$. The figure shows that for $0 \leq \gamma < \gamma^*$ the line and the curve h_1^{0+}, intersect only in one point, i.e. $\chi_\gamma = \{\omega_A^{0+}\}$. This means that, when $0 \leq \gamma < \gamma^*$, the maximum delay uncertainty δ is given by the distance between the curve h_2^{1+} and the curve h_1^{0+} for $\omega = \omega_A^{0+}$, i.e. $\delta(\gamma) = h_2^{1+}(\omega_A^{0+}) - h_1^{0+}(\omega_A^{0+})$. The situation drastically changes when $\gamma = \gamma^*$: for this value of γ the line becomes tangent to $h_1^{1-}(\omega)$ and thus another solution appears, i.e. $\chi_{\gamma^*} = \{\omega_A^{0+}, \omega_A^{1-}\}$. In order to compute γ^* we find the minimum of the function $h_1^{1-}(\omega)$. By computing $dh_1^{1-}(\omega)/d\omega = 0$ we obtain:

$$2\pi - \arctan(\omega) - \arccos\left(\frac{1}{2}\sqrt{\omega^2+1}\right) + \frac{\omega}{\omega^2+1} - \frac{\omega^2}{\sqrt{\omega^2+1}\sqrt{3-\omega^2}} = 0 \quad (18)$$

which has a unique solution for $\omega_A^{1-} \simeq 1.7114$ rad/s.

By plugging ω_A^{1-} into (13) it turns out $\gamma^* = h_1^{1-}(\omega_A^{1-}) \simeq 2.9844$. Moreover, by solving $h_1^{0+}(\omega) = \gamma^*$ the unique solution $\omega_A^{0+} = 0.2594$ rad/s is obtained and the maximum delay uncertainty at the left of γ^*, i.e. $\delta(\gamma_-^*)$, can be found as follows:

$$\delta(\gamma_-^*) = h_2^{1+}(\omega_A^{0+}) - h_1^{0+}(\omega_A^{0+}) \simeq 4.1856.$$



(a) $\alpha = 0.1$ (b) $\alpha = 1$

Fig. 4: Stability crossing curve in the case of the modified Smith predictor for two values of α

Finally, it is simple to show that

$$\delta(\gamma_+^*) = |h_2^{1-}(\omega_A^{1-}) - h_1^{1-}(\omega_A^{1-})| = |h_2^{1-}(\omega_A^{1-}) - \gamma^*|. \quad (19)$$

Thus, by plugging ω_A^{1-} in (19), it turns out $\delta(\gamma_+^*) \simeq 1.6783$. Fig. 3 shows how $\delta(\gamma_-^*)$ and $\delta(\gamma_+^*)$ can be computed.

B. The modified Smith predictor ($\alpha > 0$)

The modified Smith predictor proposed in [7] adds $\hat{b}(t)$, the output of the disturbance estimator, to the Smith predictor action $r_r(t)$, resulting in the control law $r(t) = r_r(t) + \hat{b}(t)$ (see Fig. 1). In [7] authors show that, assuming a perfect knowledge of the delay, i.e. $\Delta = 0$, the closed-loop system is asymptotically stable as long as $0 \leq k_0 < \frac{\pi}{2\overline{\tau}}$. This means that, unlike the standard Smith predictor (see Proposition 1), a delay-independent bound on the uncertainty, similar to that of Proposition 1, cannot be established.

In [7] the authors proposed to tune the additional parameter k_0 of the disturbance estimator as $k_0 = 1/(2\overline{\tau})$. The paper does not provide a robust stability analysis with respect to the delay uncertainty. Here we generalize the tuning rule proposed in [7], by setting $k_0 = \beta/\overline{\tau}$ for $\beta \in (0, \pi)$, and we compute the bound on $k\Delta$ to retain the stability as function of $k\overline{\tau}$ and β.

For a fixed $\beta \in (0, \pi)$, we denote with $\delta_\beta(\alpha)$ the maximum delay uncertainty $k\Delta$ that can be allowed to retain asymptotic stability. By considering the characteristic equation (11) for a fixed value of $\alpha > 0$, and solving the inequalities (3) and (4), it is possible to show that the frequency crossing set is $\Omega_\alpha = [\omega_a(\alpha), \omega_b(\alpha)]$ and that the stability crossing curves are a series of spiral-like curves [15]. By using the same approach shown in Section IV-A, and for a fixed α, we are able to draw the stability crossing curves (SCC) in the $k\overline{\tau}, k\Delta$ parameter space. Fig. 4 (a) and Fig. 4 (b) show the SCC in the case of $\alpha = 0.1$ and $\alpha = 1$ respectively. Both the figures show a vertical line with abscissa $k\overline{\tau} = \beta/\alpha$. Since $\alpha = k_0/k$, it turns out that $k\overline{\tau} = \beta k/k_0$ which gives $k_0 = \beta/\overline{\tau}$, that is the tuning rule we are employing for k_0. This means that $\delta_\beta(\alpha)$ is the minimum distance between the point $(\beta/\alpha, 0)$ and a generic SCC along the vertical line with abscissa β/α.

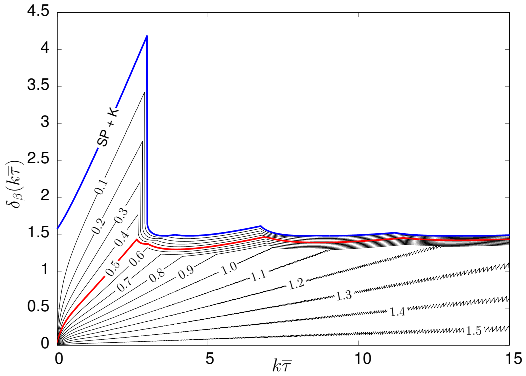


Fig. 5: The maximum delay uncertainty $\delta_\beta(k\overline{\tau})$ for $\beta = \{0.1, 0.2, \ldots, 1.5\}$

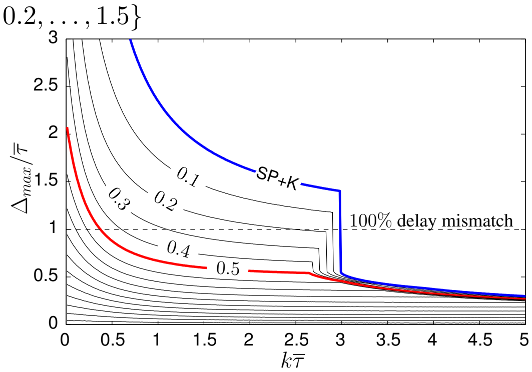


Fig. 6: Maximum relative mismatch as function of $k\overline{\tau}$ for $\beta = \{0.1, 0.2, \ldots, 1.5\}$

To show the effect of β on $\delta_\beta(\alpha)$, let us consider Fig. 4 (a): the vertical line is shown for $\beta = 0.5$, which is the tuning rule proposed in [7]; it is clear that, when β increases, the line moves to the right and eventually, for values of β close to π, $\delta_\beta(\alpha)$ decreases. This situation is more evident in the case of $\alpha = 1$ (Fig. 4 (b)) where increasing β results in decreasing $\delta_\beta(\alpha)$ almost linearly. It is possible to numerically compute $\delta_\beta(\alpha)$ as follows: for a fixed $\beta \in (0, \pi)$, we let α to vary in $(0, \overline{\alpha})$ and, by using the same geometric arguments we have described in Section IV-A, we can compute $\delta_\beta(\alpha)$. Then, once $\delta_\beta(\alpha)$ has been computed, it is possible to easily get $\delta_\beta(k\overline{\tau})$ by observing that $k\overline{\tau} = \frac{\beta}{\alpha}$.

Fig. 5 shows the curves $\delta_\beta(k\overline{\tau})$, marked with the value of β, for $\beta \in \{0.1, 0.2, \ldots, 1.5\}$ and for $k\overline{\tau} \in [0, 15]$. The figure also shows the maximum delay uncertainty of the standard Smith predictor, marked with the label "SP+K". The figure shows that the standard Smith predictor is more robust with respect to the modified Smith predictor proposed in [7]. Also, as expected, the robustness of the modified Smith predictor degrades when β increases, i.e. when the disturbance estimator is made faster.

To conclude this section, we consider the robust stability with respect to relative mismatches, i.e. $\Delta/\overline{\tau}$, which are particularly important in the case of congestion control for data networks, since the delay mismatch is due to queuing time. The maximum queuing time can be modelled as $T_q \simeq B/b$, where B is the maximum queue length (in bytes) and b is the capacity of the bottleneck link (in bytes/s) [10]. Typically, the maximum queue length B is tuned using the *bandwidth delay product* rule of thumb, i.e. $B = b \cdot \overline{\tau}$. Using such tuning rule the maximum queuing time turns out to be $T_q = \overline{\tau}$, and thus the maximum relative delay mismatch $\Delta/\overline{\tau} = 1$. Thus, it is important to tune the parameters k_0 and k such that the closed loop system is asymptotically stable for $\Delta/\overline{\tau} \leq 1$.

TABLE I: Maximum $k\overline{\tau}$ as function of β to ensure stability *wrt* 100% relative delay mismatch

β	0.1	0.2	0.3	0.4	0.5	0.6	0.7
$k\overline{\tau}$	2.90	2.49	1.09	0.6	0.37	0.23	0.11

Fig. 6 shows the maximum relative mismatch $\Delta_{max}/\overline{\tau}$ to retain stability in the case of 1) the standard Smith predictor, marked as "SP+K", and 2) the modified Smith predictor[1] [7] for $\beta \in \{0.1, 0.2, \ldots, 1.5\}$. The dashed line in the figure represents the 100% delay mismatch: it is clear that when β increases, i.e. the disturbance is rejected faster, the maximum $k\overline{\tau}$ value that can be used to be robust to a 100% delay mismatch decreases, which means that k has to decrease, which results in a slower step-response.

Table I shows the maximum value of $k\overline{\tau}$ as function of β to ensure stability with respect to 100% relative delay mismatch[2]: the tuning rule proposed in [7], i.e. $k_0 = 1/(2\overline{\tau})$, imposes to use a maximum value of $k = 0.37/\overline{\tau}$, whereas the standard Smith predictor ensures $\Delta_{max}/\overline{\tau} > 1.4025$ when $k < 2.9844/\overline{\tau}$.

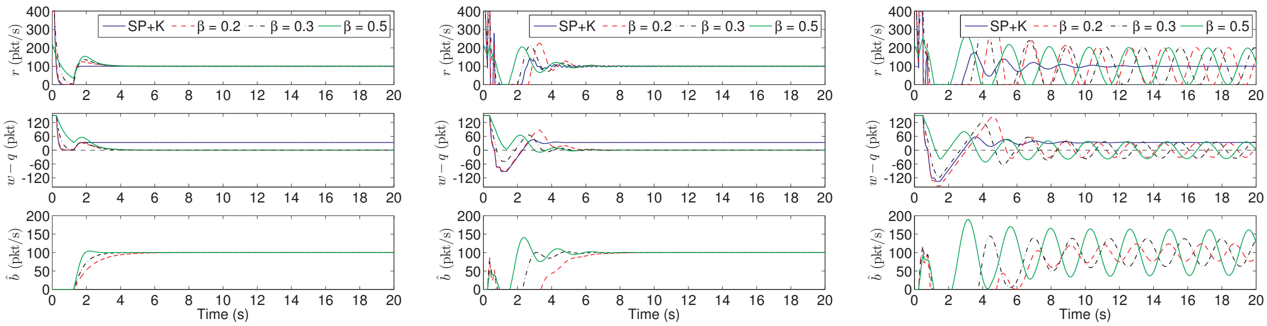
V. Numerical results

In this Section we provide the simulation results obtained by employing a Matlab SIMULINK model of Fig. 1, where two saturation blocks have been included to avoid that both the input rate $r(t)$ and the queue length $q(t)$ get negative. The set-point queue $w(t)$ is a step function starting at $t = 0$s and with a final value of 150packets. The available bandwidth $b(t)$ varies as a step-function starting at $t = 1$s, and having a final value equal to 100packet/s. The nominal RTT of the connection is $\overline{\tau} = 0.25$s. We consider a constant relative delay uncertainty $\Delta/\overline{\tau}$ equal to either 0%, 50%, or 100%. The gain k in the case of the standard Smith predictor (SSP) has been set equal to $k = \gamma^*/\overline{\tau} \simeq 11.938\text{s}^{-1}$ which is the maximum value of k ensuring robustness up to 140.25% relative delay mismatch. It is worth noting that it is not possible to find a value of k which ensures a maximum relative delay mismatch equal to 100% since, as it is shown in Fig. 6, $\Delta_{max}/\overline{\tau}$ is discontinuous at $k\overline{\tau} = \gamma^*$ and jumps from 1.4025 to 0.5479.

In the case of the modified Smith predictor (MSP), the disturbance estimator gain k_0 employs the tuning rule $k_0 = \beta/\overline{\tau}$ with three values for β, i.e. 0.2, 0.3, and 0.5. In this case, the gain k has been tuned for each of the considered value of β by means of Table I which ensures robustness up to 100% relative delay mismatch. Table II summarizes the tuned parameters used in the simulations.

[1] The tuning rule proposed in [7], corresponds to the curve marked with 0.5 in Fig. 6.

[2] The table does not consider the case with $\beta \in \{0.8, 0.9, \ldots, 1.5\}$ since for such values of β it does not exist a value of $k\overline{\tau}$ such that the maximum relative mismatch is greater than 100%.



(a) $\Delta/\overline{\tau} = 0$ (b) $\Delta/\overline{\tau} = 0.5$ (c) $\Delta/\overline{\tau} = 1$

Fig. 7: Input rate $r(t)$, error $w(t)-q(t)$, and estimated disturbance $\hat{b}(t)$, in the case of the standard Smith predictor ("SP+K") and the modified Smith predictor with $\beta \in \{0.2, 0.3, 0.5\}$

TABLE II: k and k_0 used in the simulations

Gains \ β	0 (SP+k)	0.2	0.3	0.5
k	11.938	9.96	4.36	1.48
k_0	0	0.8	1.2	2

Fig. 7 shows the input rate $r(t)$, the error $w(t)-q(t)$, and the estimated disturbance $\hat{b}(t)$, obtained in the case of $\Delta/\overline{\tau}$ equal to 0% (Fig 7 (a)), 50% (Fig. 7 (b)), 100% (Fig. 7 (c)).

Fig. 7 (a) shows the results in the case of zero delay mismatch: the input rate $r(t)$ in the case of the standard Smith predictor ("SP+K") reaches the steady-state without overshoots since the closed loop system is equivalent to a first-order system with time constant $1/k$ delayed of $\overline{\tau}$ seconds. The figure also shows that, as expected, the disturbance $b(t)$ is not rejected and a positive steady state error is present. Regarding the MSP, the figure shows that with higher values of β the estimated disturbance $\hat{b}(t)$ tracks $b(t)$ faster, at the expense of larger overshoots of the input rate $r(t)$. Finally, the SSP provides a settling time much lower with respect to the one obtained by the MSP. Fig. 7 (b) shows that, when the relative delay mismatch is 50%, oscillations of $r(t)$ get more pronounced, with the SSP controller providing a smaller overshoot and a lower settling time. Fig. 7 (c) shows that when $\Delta/\overline{\tau} = 1$, the closed loop system in the case of the modified Smith predictor is marginally stable. This is expected, since we have tuned k and k_0 to ensure asymptotic stability for $\Delta/\overline{\tau} < 1$. In the case of the SSP, the system is still asymptotically stable since, as we already mentioned, the tuned gain provides asymptotic stability up to 140.25%.

VI. Conclusions

In this paper we have considered the modified Smith predictor (MSP) proposed in [7] as a candidate for the implementation of a congestion control algorithm for computer networks which is able to reject the disturbance, i.e. the bottleneck available bandwidth, at steady state. By employing the geometric approach proposed in [15], we have quantitatively assessed the trade-off between disturbance rejection and robustness of the controller in the face of delay uncertainty.

In particular, for the standard Smith predictor (SSP) it is possible to establish a delay-independent necessary and sufficient condition that can be used to tune the proportional gain k, but the disturbance cannot be rejected; on the other hand, the MSP rejects the disturbance at steady state at the expense of being less robust with respect to the SSP. Finally, a guideline for the tuning of the additional parameter k_0 used in the disturbance estimator is also given.

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